ME 314 - Engineering Design : Mechanical Components
Lecture 4

Note Title

Chapter 4 - Stress, Strain, and Deflection

The stress & strain states at critical points in a component are required to determine whether the component will satisfy the design requirements for strength and deflection.

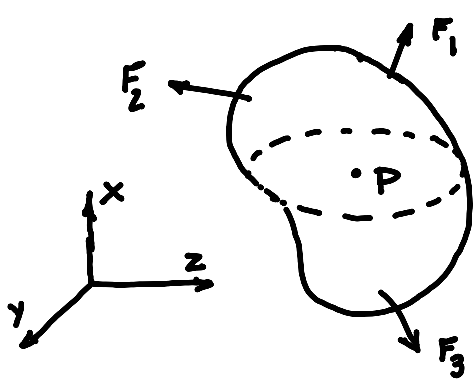
In Chapter 3, we reviewed the concepts of equilibrium and motion that are covered in Statics (EM 200) and Dynamics (EM 220) and analyzed the loads acting on various elements by considering the Free Body Diagram (FBD).

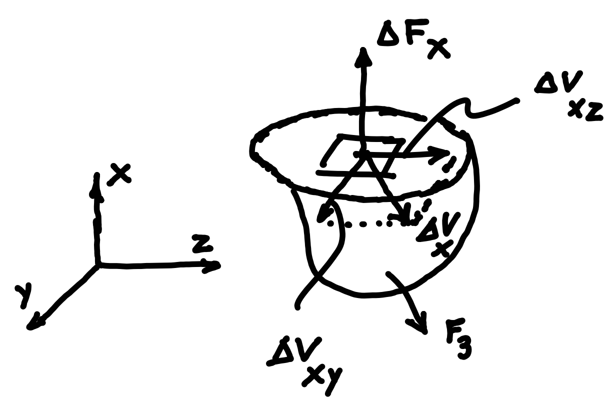
The objective of Chapter 4 is to review the concepts of stress, strain, and deflection that are covered in Mechanics of Materials (ME 304) and the course Materials, Manufacturing, and Design (ME 340).

4.1 Stress

We are interested in characterizing the intensity of forces at a point inside of a component that is under a set of external loads.

To this end, consider a machine component of arbitrary shape as shown in the figure.

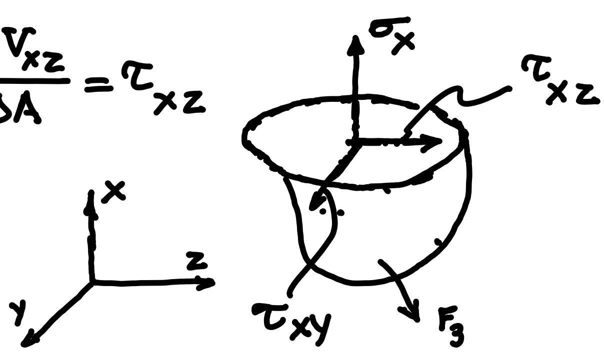




To find the state of stress (i.e., intensity of forces) at point P, we pass a plane parallel to the yz-plane through P and consider the normal and shearing forces acting on a small area around P on the cross-section formed by the passing plane. Note that while the normal force ΔF_x is parallel to the x-axis, the shearing force ΔV_x, can have any direction. Resolve ΔV_x into two vectors parallel to y- and z-axes. Denote them by ΔV_{xy} and ΔV_{xz}, respectively. Divide these forces by ΔA and let ΔA approach zero, to find

$$\lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A} = \sigma_x \quad , \quad \lim_{\Delta A \to 0} \frac{\Delta V_{xz}}{\Delta A} = \tau_{xz}$$

$$\lim_{\Delta A \to 0} \frac{\Delta V_{xy}}{\Delta A} = \tau_{xy}$$



These are the definitions of the three (Cartesian) components of stress σ_x, τ_{xy}, and τ_{xz} acting on a plane parallel to the yz-plane and passing through the point P. Note that the first index indicates the direction of the normal to the plane (in this case x) and the second index indicates the direction along which the stress acts.

If we pass a section through P parallel to the xz-plane, we arrive at three components τ_{yx}, σ_y, and τ_{yz}. Similarly, a section through P parallel to the xy-plane leads to the three components τ_{zx}, τ_{zy}, and σ_z.

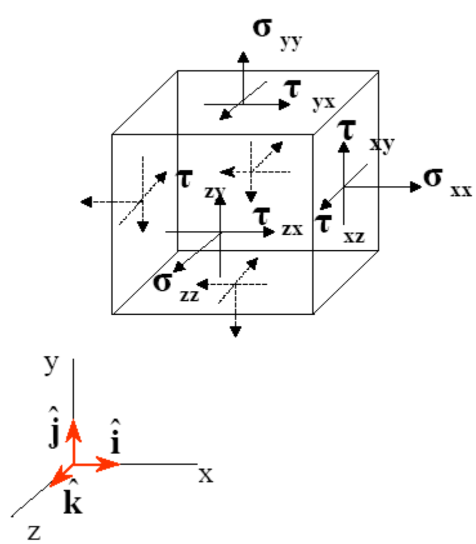
In summary, there are nine components of stress which act on three mutually orthogonal planes. For clarity, these are represented on a cube centered at point P as shown in figure below. The stresses acting on the faces of the cube are slightly different from the actual stresses at P. Here, we assume the difference is negligible.

Equilibrium of the cube requires that $\Sigma F = 0$ and $\Sigma M_o = 0$. Balance of moments about the z-axis yields

$$\overset{+}{\curvearrowleft} \Sigma M_z = 0 \qquad \tau_{xy}(dy)(dz)\left(\frac{dx}{2}\right) - \tau_{yx}(dx)(dz)\left(\frac{dy}{2}\right) = 0$$

$$\text{or} \quad \tau_{xy} = \tau_{yx}$$

3D Stress Components



Note that the tensor sign convention is used.

There are nine components of stress.

Moment equilibrium can be used to reduce the number of stress components to six.

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{xz} = \tau_{zx}$$

$$\tau_{yz} = \tau_{zy}$$

Similarly, $\Sigma M_y = 0$ and $\Sigma M_x = 0$ yield $\tau_{xz} = \tau_{zx}$ and $\tau_{yz} = \tau_{zy}$. **Thus only 6 components of stress are independent. The array of stress components can be represented by a 3x3 matrix, σ, which can be shown to be a *tensor* of rank two.**

Cauchy Stress Tensor

Tensor Transformation Equation

$$\underset{\approx}{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

$\underset{\approx}{\boldsymbol{\sigma}}$ is known as the Cauchy stress tensor. Its Cartesian components are shown written in matrix form.

Tensors are quantities that are invariant to a coordinate transformation.

Tensor Transformation Equation

$$\sigma_{mn} = \beta_{mi}\,\sigma_{ij}\,\beta_{jn}$$

A vector is an example of a first order tensor. It can be written with respect to many different coordinate systems.

In 1822, Cauchy, the French engineer and mathematician, introduced the above concept of stress and showed that given its nine components, the force per unit area, or traction,

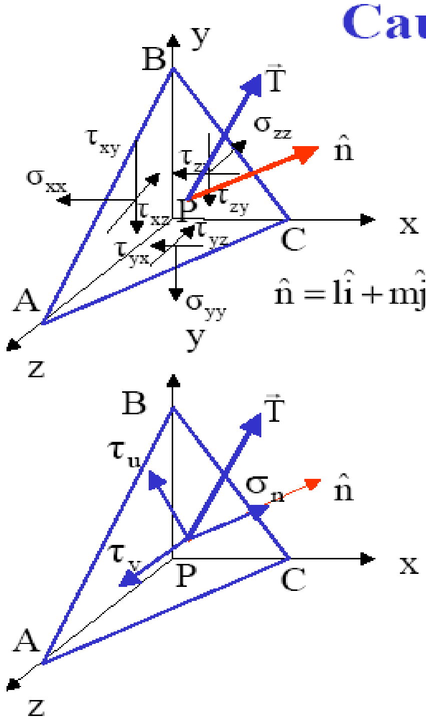
$$\mathbf{T} = T_x\,\mathbf{i} + T_y\,\mathbf{j} + T_z\,\mathbf{k}$$

on any plane with unit normal

$$\mathbf{n} = l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k}$$

can be calculated.

Cauchy Formula



$$\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$$

ΣF in the x,y,and z directions yields the Cauchy Stress Formula.

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix}$$

This equation is similar to the Mohr's circle transformation-of-axis equation

Hence, the (Cauchy) stress σ characterizes the intensity of forces at a point P. Given σ, we can find the traction on any plane with direction cosines l, m, **and** n**:**

$$T_x = \sigma_x l + \tau_{xy} m + \tau_{xz} n$$

$$T_y = \tau_{yx} l + \sigma_y m + \tau_{yz} n$$

$$T_z = \tau_{zx} l + \tau_{zy} m + \sigma_z n$$

This is referred to as Cauchy's stress formula.

Note that T is not along n, in general, as it would be for a fluid at rest. Solids can sustain shear in equilibrium. It turns out that for solids there are particular planes called "principal planes" for which T would be along n (just like fluids at rest).

We would like to find these principal planes and the principal stresses that act on these planes because they happen to be the extremal values of normal stresses.

3D Principal Stresses

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix}$$

The shear stress on planes normal to the principal stress directions are zero.

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \sigma \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}$$

We need to find the plane in which the stress is in the direction of the outward unit normal.

$$\begin{bmatrix} (\sigma_{xx} - \sigma) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_{yy} - \sigma) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_{zz} - \sigma) \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

This is a homogeneous linear equation.

$l = \frac{0}{\Delta}$

$m = \frac{0}{\Delta}$

$n = \frac{0}{\Delta}$

3D Principal Stresses
(Eigenvalue Problem)

$$\begin{bmatrix} (\sigma_{xx} - \sigma) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_{yy} - \sigma) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_{zz} - \sigma) \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

A homogeneous linear equation has a solution only if the determinant of the coefficient matrix is equal to zero.

$$\begin{vmatrix} (\sigma_{xx} - \sigma) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_{yy} - \sigma) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_{zz} - \sigma) \end{vmatrix} = 0$$

This is an eigenvalue problem.

3D Principal Stresses

(Characteristic Equation)

$$\begin{vmatrix} (\sigma_{xx}-\sigma) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_{yy}-\sigma) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_{zz}-\sigma) \end{vmatrix} = 0$$

The determinant can be expanded to yield the equation

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

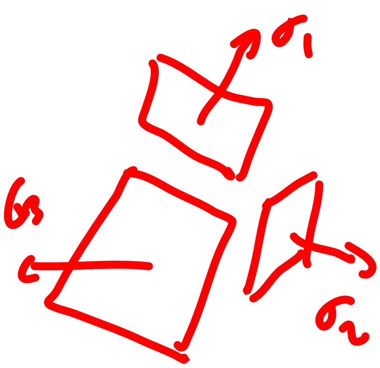
I_1, I_2, and I_3 are known as the first, second, and third invariants of the Cauchy stress tensor.

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2$$

3D Principal Stresses



$\sigma_1 \neq \sigma_2 \neq \sigma_3$

Characteristic Equation

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

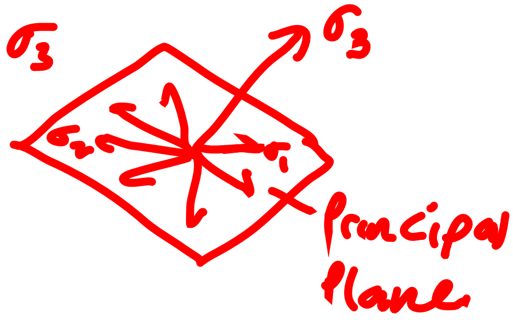
There are three roots to the characteristic equation, σ_1, σ_2, and σ_3.

Each root is one of the principal stresses.

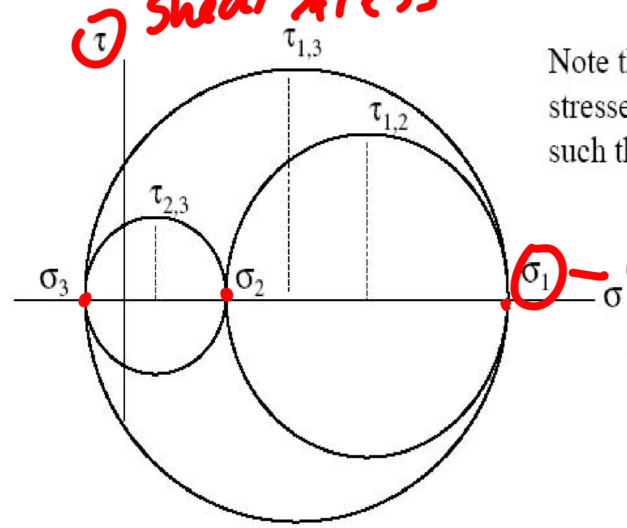
The direction cosines define the principal directions or planes.

The direction cosines can be found by substituting the principal stresses into the homogeneous equation and solving.

$\sigma_1 = \sigma_2 \neq \sigma_3$



3D Mohr's Circles (Review)



Note that the principal stresses have been ordered such that $\sigma_1 \ge \sigma_2 \ge \sigma_3$.

Graphical Approach

Maximum shear stresses

$$\tau_{1,2} = \frac{\sigma_1 - \sigma_2}{2}$$

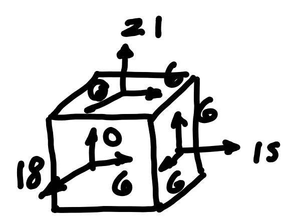
$$\tau_{2,3} = \frac{\sigma_2 - \sigma_3}{2}$$

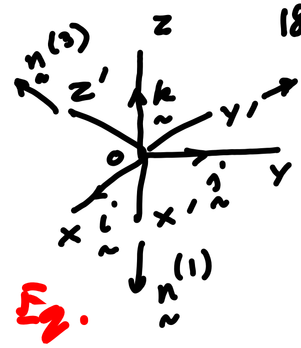
$$\tau_{1,3} = \frac{\sigma_1 - \sigma_3}{2}$$

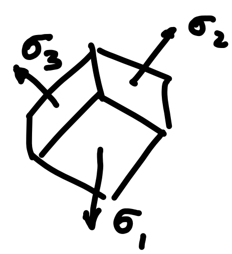
Example: **Find the principal stresses and their corresponding planes at a point P where the stress σ is given by**

$$\underset{\sim}{\sigma} = \begin{bmatrix} 18 & 6 & 6 \\ 6 & 15 & 0 \\ 6 & 0 & 21 \end{bmatrix}_{\{oxyz\}} \text{MPa}$$

$$= \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}_{\{ox'y'z'\}}$$







Solution: The Characteristic Eq.

$$\begin{vmatrix} 18-\sigma & 6 & 6 \\ 6 & 15-\sigma & 0 \\ 6 & 0 & 21-\sigma \end{vmatrix} = 0$$

$$\sigma^3 - 54\sigma^2 + 891\sigma - 4374 = 0$$

Roots = 9, 18, 27

$\sigma_1 = 27$, $\sigma_2 = 18$, $\sigma_3 = 9$

$$\left.\begin{aligned} * \ (18-27)l + 6m + 6n &= 0 \\ 6l + (15-27)m + 0n &= 0 \\ 6l + 0m + (21-27)n &= 0 \end{aligned}\right\} \quad \begin{aligned} -9l + 6m + 6n &= 0 \\ 6l - 12m &= 0 \\ 6l - 6n &= 0 \\ l^2 + m^2 + n^2 &= 0 \end{aligned}$$

$l = 2m$ $\quad l^2 + \frac{l^2}{4} + l^2 = 1$ $\quad l^2 = \frac{4}{9} = \pm\frac{2}{3}$

$l = n$

$n^{(1)} = \pm\left(\frac{2}{3}i + \frac{1}{3}j + \frac{2}{3}k\right)$ (choose "+" w/o any loss in generality)

$\boxed{\text{For } \sigma_2 = 18}$ $\quad l^2 + m^2 + n^2 = 1$

$l^2 = \frac{1}{9} = \pm\frac{1}{3}$

$6m + 6n = 0$

$6l - 3m = 0 \Rightarrow 2l = m$

$6l + 3n = 0 \Rightarrow -2l = n$

$n^{(2)} = \pm\left(\frac{1}{3}i + \frac{2}{3}j - \frac{2}{3}k\right)$

$\boxed{\text{For } \sigma_3 = 9}$ $\quad n^{(3)} = n^{(1)} \wedge n^{(2)}$

$n^3 = n^{(1)} \times n^{(2)} = \left(\frac{2}{3}i + \frac{1}{3}j + \frac{2}{3}k\right) \times \left(-\frac{1}{3}i - \frac{2}{3}j + \frac{2}{3}k\right)$

$n^3 = \frac{2}{3}i - \frac{2}{3}j - k$

Note that the transformation (i.e., rotation) matrix that corresponds to the rotation of $\{Oxyz\}$ into $\{Ox'y'z'\}$ is constructed from the components of $\underset{\sim}{n}^{(1)}$, $\underset{\sim}{n}^{(2)}$, and $\underset{\sim}{n}^{(3)}$:

$$\underset{\sim}{\beta} = [\beta_{ij}] = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ -1 & -2 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

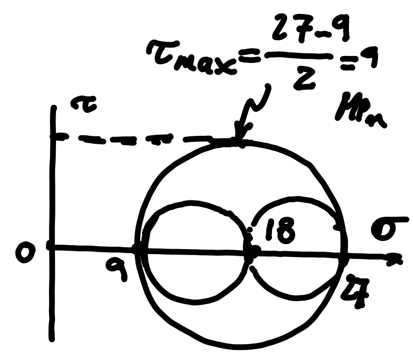
So that

$$\underset{\sim}{\sigma}' = \underset{\sim}{\beta}\,\underset{\sim}{\sigma}\,\beta^{T}$$

You can verify that

$$\begin{bmatrix} 27 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 2 & 1 & 2 \\ -1 & -2 & 2 \\ 2 & -2 & -1 \end{bmatrix}\begin{bmatrix} 18 & 6 & 6 \\ 6 & 15 & 0 \\ 6 & 0 & 21 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ 1 & -2 & -2 \\ 2 & 2 & -1 \end{bmatrix}$$

Mohr Circles are as shown.



Summarizing:

$$\underset{\sim}{\sigma} = \begin{bmatrix} 18 & 6 & 6 \\ 6 & 15 & 0 \\ 6 & 0 & 21 \end{bmatrix}_{\{Oxyz\}} \text{MPa}$$

$$= \begin{bmatrix} 27 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 9 \end{bmatrix}_{\{Ox'y'z'\}} \text{MPa}$$